

Discrete Translates in $L^p(\mathbb{R})$

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Abstract

A set $\Lambda \subset \mathbb{R}$ is called p -spectral if there is a function $\varphi \in L^p(\mathbb{R})$ whose Λ -translates $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$. We prove that exponentially small non-zero perturbations of the integers are p -spectral sets for all $p > 1$.

1 Introduction

1. Completeness of translates. In what follows we will use the standard form of Fourier transform

$$\varphi(x) = \hat{\Phi}(x) := \int_{\mathbb{R}} e^{-2\pi i t x} \Phi(t) dt.$$

Classical Wiener's theorems [W32] provide necessary and sufficient conditions on a function φ whose translates $\{\varphi(t - s), s \in \mathbb{R}\}$ span the space $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$:

- (i) The translates of $\varphi \in L^1(\mathbb{R})$ span $L^1(\mathbb{R})$ if and only if $\hat{\varphi}$ does not vanish;
- (ii) The translates of $\varphi \in L^2(\mathbb{R})$ span $L^2(\mathbb{R})$ if and only if $\hat{\varphi}$ is non-zero almost everywhere on \mathbb{R} .

When $1 < p < 2$, Beurling [B51] proved that the translates of a function $\varphi \in L^p(\mathbb{R}) \cap L^1(\mathbb{R})$ span $L^p(\mathbb{R})$ provided the Hausdorff dimension of the zero set of $\hat{\varphi}$ is less than $2(p - 1)/p$. However, this

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condition is not necessary. Moreover, the spanning property of the translates of $\varphi \in L^p(\mathbb{R})$ cannot be characterized in terms of the zero set of $\hat{\varphi}$, see [LO11].

2. Discrete spectra. It is well known that sometimes even a *discrete* set of translates may span $L^p(\mathbb{R})$.

Definition 1. *We say that a discrete set $\Lambda \subset \mathbb{R}$ is a p -spectrum (for translates), if there is a function $\varphi \in L^p(\mathbb{R})$ such that the family of translates*

$$\{\varphi(t - \lambda), \lambda \in \Lambda\}$$

spans $L^p(\mathbb{R})$. Such a function φ is called a Λ -generator.

For $p = 2$, using the Fourier transform one can re-formulate the definition above as follows:

A set Λ is a 2-spectrum if and only if there exists $\Phi \in L^2(\mathbb{R})$ such that the system

$$\{\Phi(t)e^{i\lambda t}, \lambda \in \Lambda\}$$

spans the whole space $L^2(\mathbb{R})$.

Using this, it is easy to see that the set of integers \mathbb{Z} is not a 2-spectrum. On the other hand, the following is true:

Every sequence Λ obtained by a "perturbation" of integers,

$$\Lambda = \{\lambda_n := n + a_n, n \in \mathbb{Z}\}, \quad a_n \neq 0, n \in \mathbb{Z}, \quad (1)$$

where $a_n \rightarrow 0$ as $|n| \rightarrow \infty$, is a 2-spectrum ([O197]).

Assume the "perturbations" a_n are exponentially small:

$$|a_n| < Cr^{|n|}, \quad n \in \mathbb{Z}; \quad \text{for some } C > 0, 0 < r < 1. \quad (2)$$

Then a stronger result is true ([OU04]):

If Λ satisfies (1) and (2), then a Λ -generator φ can be chosen to be a Schwartz function. The decay condition (2) is essentially sharp.

This result may seem surprising, since when (2) holds, the set Λ is "very close" to the "limiting case" $\Lambda = \mathbb{Z}$ when no generator exists.

Since every p_0 -spectrum Λ is also a p -spectrum for every $p > p_0$ (see [Bl06]), the result above holds for $p > 2$. However, in fact no perturbations are "needed" in this case:

The set of integers \mathbb{Z} is a p -spectrum for all $p > 2$, see [AO96].

On the other hand, for $p = 1$ the spectrum is never uniformly discrete. The following is proved in [BOU06]:

A set Λ is a 1-spectrum if and only if the Beurling–Malliavin density of Λ is infinite.

Notice, so far $p = 1$ is the only case when an effective characterization of p -spectra is known.

A detailed survey on the topic, including the results above, may be found in [OU16].

3. The case $1 < p < 2$. This case is much less investigated. In particular, it has been an open problem whether a p -spectral set can be uniformly discrete, or at least have a finite density.

In this note we give a positive answer to this question. Our main result is the following

Theorem 1. *Every set Λ satisfying (1) and (2) is a p -spectrum, for every $p > 1$.*

We will construct a function φ , which is a Λ -generator for every Λ satisfying (1) and (2) and all $p > 1$ simultaneously. In addition, φ will have the nice properties of analyticity and fast decay. The construction is based on a uniqueness theorem for a certain class of tempered distributions.

2 Functions with Deep Zeros

In this section we present a uniqueness result for a class of analytic functions whose inverse Fourier transforms have "deep zeros". Then in the next section this result will be extended to a certain class of temperate distributions.

Definition 2. Denote by K_0 the class of continuous functions Φ satisfying the condition

$$|\Phi(t)| \leq C_1 e^{-C_2(|t| + \frac{1}{d(t, \mathbb{Z})})}, \quad t \in \mathbb{R}, \quad (3)$$

where $d(t, \mathbb{Z}) = \min_{n \in \mathbb{Z}} |t - n|$, and $C_1 = C_1(\Phi)$, $C_2 = C_2(\Phi)$ are positive constants depending on Φ .

Condition (3) means that Φ has "deep" zeros on the set of integers and at infinity.

Next, let \hat{K}_0 denote the class of the Fourier transforms $\varphi = \hat{\Phi}$ of functions $\Phi \in K_0$.

Condition (3) implies that Φ has an exponential decay at $\pm\infty$. Hence, every element $\varphi \in \hat{K}_0$ is analytic and bounded in the horizontal strip $|\operatorname{Im} z| < C$, $0 < C < 1/C_2$, where C_2 is the constant in (3). Clearly, every derivative of φ also belongs to \hat{K}_0 .

Model Example. Suppose Λ satisfies (1) and (2) and $\varphi \in \hat{K}_0$. If $\varphi|_\Lambda = 0$, then $\varphi = 0$.

In other words, the sets Λ satisfying (1) and (2) are uniqueness sets for the class \hat{K}_0 .

Sketch of Proof. Assume $\varphi \in \hat{K}_0$ and $\varphi|_\Lambda = 0$ for some Λ satisfying (1) and (2). We have to show that $\varphi = 0$.

Since φ' is bounded on \mathbb{R} , from (1) and (2) we get

$$|\varphi(n)| = O(r^{|n|}), \quad |n| \rightarrow \infty, \quad (4)$$

where $0 < r < 1$.

Consider the periodization $P(\Phi)$ of Φ defined by

$$P(\Phi)(t) := \sum_{k \in \mathbb{Z}} \Phi(t + k).$$

The Fourier decomposition of $P(\Phi)$ has the form

$$P(\Phi)(t) = \sum_{n \in \mathbb{Z}} \varphi(n) e^{2\pi i n t}.$$

By (4), $P(\Phi)$ is analytic in some horizontal strip containing \mathbb{R} . On the other hand, it easily follows from (3) that it has zero of infinite order at the origin. Hence, $P(\Phi) = 0$, so that we have

$$\varphi(n) = 0, \quad n \in \mathbb{Z}.$$

Iterating this argument, one prove that all derivatives of φ vanish at the integers, and the result follows. See details in [OU16], Lecture 11.

□

3 Distributions with Deep Zeros

Let $S(\mathbb{R})$ denote the space of Schwartz test-functions, and $S'(\mathbb{R})$ the space of tempered distributions on \mathbb{R} . We will denote by $\langle F, \Phi \rangle$ the action of the distribution $F \in S'(\mathbb{R})$ on the test function $\Phi \in S(\mathbb{R})$.

We would like to extend condition (3) to tempered distributions.

Recall that every tempered distribution $F \in \mathcal{S}'(\mathbb{R})$ is the distributional derivative of finite order, $F = \Phi^{(k)}$, of a continuous function Φ having at most polynomial growth on the real axis (see [FJ98], Theorem 8.3.1).

Definition 3. Denote by K the class of all tempered distributions $F \in \mathcal{S}'(\mathbb{R})$ which admit a representation

$$F = \Phi_1^{(k_1)} + \dots + \Phi_l^{(k_l)}, \quad (5)$$

where $l \in \mathbb{N}$, $k_1, \dots, k_l \in \mathbb{N} \cup \{0\}$ and Φ_1, \dots, Φ_l are continuous functions satisfying (3).

Let \hat{K} denote the class the distributional Fourier transforms of the elements of K .

Assume Φ satisfies (3). Then the Fourier transform $\varphi = \hat{\Phi}$ is analytic and bounded in some horizontal strip. The distributional Fourier transform of $\Phi^{(k)}$ is the function

$$\widehat{\Phi^{(k)}} = (2\pi ix)^k \varphi(x).$$

It is therefore an analytic function of at most polynomial growth in the strip.

By the definition of class \hat{K} , we have

$$x^k \varphi(x) \in K, \quad \text{for every } k \geq 0. \quad (6)$$

From (5) it follows that \hat{K} consists of the functions f admitting a representation

$$f(x) = \sum_{j=1}^l (2\pi ix)^{k_j} \varphi_j(x), \quad \varphi_j = \hat{\Phi}_j, \Phi_j \in K_0, \quad j = 1, \dots, l.$$

We will now list several simple properties of the class \hat{K} .

Lemma 1. (a) \hat{K} is a linear space over \mathbb{C} .

(b) Every element $f \in \hat{K}$ is analytic and of at most polynomial growth in some strip $|\operatorname{Im} z| < C$, $C = C(f) > 0$.

(c) If $f \in \hat{K}$, then $xf(x) \in \hat{K}$ and $f' \in \hat{K}$.

(d) If $f \in \hat{K}$ then $\operatorname{Re} f(x) \in \hat{K}$ and $\operatorname{Im} f(x) \in \hat{K}$.

(e) Assume $\varphi = \hat{\Phi}$, where $\Phi \in \mathcal{S}(\mathbb{R})$ is such that $\Phi, \Phi' \in K_0$. Then for every $1 \leq q \leq \infty$ and every $h \in L^q(\mathbb{R})$ we have $\varphi * h \in \hat{K}$.

Proof. Statements (a) and (b) are obvious.

Clearly, it suffices to prove statements (c) and (d) for the case $l = 1$ in (5), i.e. when $f(x) = \widehat{\Phi^{(k)}} = (2\pi ix)^k \varphi(x)$.

By (6), we see that $xf(x) \in K$.

Since the function $t\Phi(t)$ also satisfies (3), then

$$\varphi' = (-2\pi it)\widehat{\Phi}(t) \in \hat{K}.$$

From (6) we get $x^k \varphi'(x) \in \hat{K}$ and $x^{k-1} \varphi(x) \in \hat{K}$ which yields $f' \in \hat{K}$. This proves (c).

Now, write

$$\Phi(t) := \Phi_r(t) + i\Phi_i(t) := \frac{\Phi(t) + \Phi(-t)}{2} + i\frac{\Phi(t) - \Phi(-t)}{2i}.$$

Clearly, Φ_r, Φ_i satisfy (3). So, we have $\varphi = \varphi_r + i\varphi_i$, where the functions $\varphi_r = \hat{\Phi}_r \in \hat{K}$ and $\varphi_i = \hat{\Phi}_i \in \hat{K}$ are real on the real line. Hence, if k is even, we see that

$$\operatorname{Re} f(x) = (2\pi ix)^k \varphi_r(x) \in \hat{K}, \quad \operatorname{Im} f(x) = (2\pi ix)^k \varphi_i(x) \in \hat{K}.$$

Similarly, one proves (d) for the odd k .

To prove (e), write $h(x) = h_1(x) + h_2(x)$, where

$$h_1(x) := \frac{h(x)}{1 - 2\pi ix}, \quad h_2(x) := -2\pi ix h_1(x).$$

Hence, $\varphi * h = \varphi * h_1 + \varphi * h_2$.

Clearly, $h_1 \in L^1(\mathbb{R})$, and so its inverse Fourier transform H_1 is continuous and bounded. This gives $\varphi * h_1 = \widehat{\Phi H_1} \in \hat{K}$.

Observe that $h_2 = \widehat{H'_1}$, and so the inverse Fourier transform of $\varphi * h_2$ is the distribution $\Phi H'_1 = (\Phi H_1)' - \Phi' H_1$. Since, by assumption, Φ' satisfies (3), we conclude that $\varphi * h_2 \in \hat{K}$, which completes the proof of Lemma 1. \square

4 Uniqueness Theorem for the Class \hat{K}

Theorem 2. *Assume Λ satisfies (1) and (2) and $f \in \hat{K}$. If $f|_\Lambda = 0$ then $f = 0$.*

Hence, the sets Λ satisfying (1) and (2) are uniqueness sets for the class \hat{K} .

Main Lemma. Assume Λ satisfies (1) and (2). Then

$$f \in \hat{K}, f|_{\Lambda} = 0 \Rightarrow f|_{\mathbb{Z}} = 0.$$

Proof. For convenience, in what follows we denote by C different positive constants.

Take any function $f \in \hat{K}$ which vanishes on Λ . We have to show that $f|_{\mathbb{Z}} = 0$.

By Lemma 1, $f' \in K$ and has at most polynomial growth on \mathbb{R} . So, it follows from (1) and (2) that

$$|f(n)| < C|n|^C r^{|n|}, \quad n \in \mathbb{Z},$$

where $0 < r < 1$. This shows that the function

$$R(t) := \sum_{n \in \mathbb{Z}} f(n) e^{2\pi i n t}, \quad t \in \mathbb{R}.$$

is analytic in some strip $|\operatorname{Im} z| < C$.

Clearly, to prove the lemma it suffices to show that $R(t) \equiv 0$. For this purpose we introduce the function $f_{\epsilon} := f h_{\epsilon}$, where $\epsilon > 0$ and

$$h_{\epsilon}(x) := \left(\frac{\sin(2\pi \epsilon x)}{2\pi \epsilon x} \right)^N.$$

We will assume that N is an even integer so large that $f_{\epsilon} \in L^1(\mathbb{R})$.

It is easy to see that $h_{\epsilon} = \hat{H}_{\epsilon}$, where

$$H_{\epsilon}(t) = \left(\frac{1}{2\epsilon} 1_{\epsilon}(t) \right)^{N*}.$$

Here 1_{ϵ} is the indicator function of $[-\epsilon, \epsilon]$. Hence, $H_{\epsilon}(t) \geq 0, t \in \mathbb{R}$, and so

$$\|H_{\epsilon}\|_1 = h_{\epsilon}(0) = 1.$$

Given $d > 0$, we denote by $S(d)$ the subspace of $S(\mathbb{R})$ of functions Ψ vanishing outside $(-d, d)$. In what follows we always assume that $d, \epsilon < 1/4$, so that $d + \epsilon < 1/2$.

Fix any function $\Psi \in S(-d, d)$. Then

$$\Psi * H_{\epsilon} \in S(d + N\epsilon),$$

and we have

$$\|(\Psi * H_{\epsilon})^{(k)}\|_{\infty} \leq \|\Psi^{(k)}\|_{\infty} \|H_{\epsilon}\|_1 = \|\Psi^{(k)}\|_{\infty}. \quad (7)$$

Denote by F_ϵ the inverse Fourier transform of f_ϵ , and by ψ the Fourier transform of Ψ . From (5), for every $n \in \mathbb{Z}$ we obtain:

$$\begin{aligned} |\langle F_\epsilon, \Psi(t-n) \rangle| &= |\langle f_\epsilon, e^{2\pi i n x} \psi \rangle| = |\langle f, e^{2\pi i n x} \psi h_\epsilon \rangle| = \\ &= |\langle \sum_{j=1}^l \Phi_j^{(k_j)}, (\Psi * H_\epsilon)(t-n) \rangle| \leq \|(\Psi * H_\epsilon)^{(k)}\|_\infty \sum_{j=1}^l \int_{n-d-N\epsilon}^{n+d+N\epsilon} |\Phi_j|, \end{aligned}$$

where $k := \max_j k_j$. Recalling that the functions Φ_j satisfy (3), by (7) we obtain

$$|\langle F_\epsilon, \Psi(t-n) \rangle| \leq C e^{-C(|n| + \frac{1}{N\epsilon+d})} \|\Psi^{(k)}\|_\infty, \quad n \in \mathbb{Z}. \quad (8)$$

Set

$$R_\epsilon(t) := \sum_{n \in \mathbb{Z}} f_\epsilon(n) e^{2\pi i n t},$$

and let us calculate the product $\langle R_\epsilon, \Psi \rangle$.

By the Poisson formula,

$$R_\epsilon(t) = \sum_{n \in \mathbb{Z}} f_\epsilon(n) e^{2\pi i n t} = \sum_{n \in \mathbb{Z}} F_\epsilon(t+n).$$

Therefore, by (8),

$$\begin{aligned} |\langle R_\epsilon, \Psi \rangle| &= |\langle \sum_{n \in \mathbb{Z}} F_\epsilon(t+n), \Psi(t) \rangle| = |\langle F_\epsilon(t), \sum_{n \in \mathbb{Z}} \Psi(t-n) \rangle| \\ &\leq C \|\Psi^{(k)}\|_\infty e^{-\frac{C}{N\epsilon+d}}. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ we arrive at the inequality:

$$|\langle R, \psi \rangle| \leq C \|\Psi^{(k)}\|_\infty e^{-\frac{C}{d}}, \quad (9)$$

which holds for some k and every $0 < d < 1/2$ and $\Psi \in S(d)$.

Now, the Main Lemma follows from

Lemma 2. *Assume a function R is analytic in a strip $|\operatorname{Im} z| < C$ and satisfies (9). Then $R = 0$.*

The proof is simple: One may use the standard calculus to check that R must have zero of infinite order at the origin, and so $R = 0$. We omit the details. \square

5 Proofs of Theorem 1 and 2

Proof of Theorem 2. Assume Λ satisfies (1) and (2), $f \in \hat{K}$ and $f|_{\Lambda} = 0$. Write $f = f_r + if_i$, where $f_r(x) := \operatorname{Re} f(x)$ and $f_i(x) := \operatorname{Im} f(x)$.

Let us show that $f_r = 0$. This function vanishes on Λ and by Lemma 1, $f_r \in K$. It follows from the Main Lemma that $f_r|_{\mathbb{Z}} = 0$.

Since f_r is real for real x and vanishes both on \mathbb{Z} and on $\Lambda = \{n + a_n, n \in \mathbb{Z}\}$, $a_n \neq 0$, its derivative f'_r vanishes on some set $\Lambda_1 := \{n + a_1^{(1)}\}$, where each point $a_1^{(1)}$ lies inside the open interval between 0 and a_n . Since Λ satisfies (1) and (2), so does Λ_1 . By Lemma 1, $f'_r \in K$. We now we apply the Main Lemma again to get $f'_r|_{\mathbb{Z}} = 0$. A straightforward iteration of this argument proves that $f_r^{(j)}|_{\mathbb{Z}} = 0$ for every $j \in \mathbb{N}$. Since f_r is analytic in some strip $|\operatorname{Im} z| < C$, we conclude that $f_r = 0$. Similarly, one proves that $f_i = 0$, and so $f = 0$. \square

Proof of Theorem 1. Choose any real function $\Phi \in S(\mathbb{R})$ such that Φ and Φ' satisfy (3) and such that $\Phi(-t) = \Phi(t) > 0$ for $t \notin \mathbb{Z}$. Then its Fourier transform φ is real and $\varphi(-x) = \varphi(x)$, $x \in \mathbb{R}$.

Let Λ satisfy (1) and (2). Assume the set of translates $\{\varphi(x - \lambda), \lambda \in \Lambda\}$ does not span $L^p(\mathbb{R})$. Then there is a non-trivial function $h \in L^q(\mathbb{R})$, $1/p + 1/q = 1$, such that

$$\int_{\mathbb{R}} \varphi(x - \lambda) h(x) dx = (\varphi * f)(\lambda) = 0, \quad \lambda \in \Lambda.$$

By Lemma 1, $\varphi * h \in \hat{K}$. Theorem 2 yields $\varphi * h = 0$. This means that all translates $\{\varphi(x - s), s \in \mathbb{R}\}$ do not span the space $L^p(\mathbb{R})$. However, this contradicts Beurling's theorem cited in the introduction. Theorem 1 is proved. \square

References

- [AO96] A. Atzmon and A. Olevskii, 'Completeness of integer translates in function spaces on \mathbb{R} ', *J. Approx. Theory* 87, (1996), no. 3, 291–327.
- [B51] A. Beurling, 'On a closure problem', *Ark. Mat.* 1, (1951). 301–303. See also: *The Collected Works of Arne Beurling*, in: Harmonic Analysis, vol. 2, Harmonic Analysis. Birkhuser, Boston, 1989.

- [Bl06] N. Blank, ‘Generating sets for Beurling algebras’, *J. Approx. Theory* 140, (2006), no. 1, 61–70.
- [BOU06] J. Bruna, A. Olevskii and A. Ulanovskii, ‘Completeness in $L^1(\mathbb{R})$ of discrete translates’, *Rev. Mat. Iberoam.* 22, (2006), no. 1, 1–16.
- [FJ98] F.J. Friedlander and M.S. Joshi, ‘Introduction to the Theory of Distributions.’ Cambridge University Press, 1998.
- [LO11] N. Lev and A. Olevskii, ‘Wiener’s ”closure of translates” problem and Piatetski–Shapiro’s uniqueness phenomenon’, *Ann. of Math.* (2) 174, (2011), no. 1, 519–541.
- [Ol97] A. Olevskii, ‘Completeness in $L^2(\mathbb{R})$ of almost integer translates’, *C. R. Acad. Sci. Paris Sér. I Math.* 324, (1997), no. 9, 987–991.
- [OU04] A. Olevskii and A. Ulanovskii, ‘Almost integer translates. Do nice generators exist?’, *J. Fourier Anal. Appl.* 10, (2004), no. 1, 93–104.
- [OU16] A. Olevskii and A. Ulanovskii, ‘Functions with Disconnected Spectrum: Sampling, Interpolation, Translates. AMS, University Lecture Series, 65, 2016.
- [W32] N. Wiener, ‘Tauberian Theorems’, *Annals of Math.* 33 (1): 1–100, 1932.